ALGEBRA

When we play games with computers we play by running, jumping and or finding secret things. Well, with Algebra we play with letters, numbers and symbols. And we also get to find secret things. Once we learn some of the ‘tricks’ it becomes a fun challenge to work with our skills in solving each puzzle. So, Algebra is all about solving puzzles. In this chapter we are going to learn some of the skills that help in solving mathematics puzzles.

Binary Operations

The Binary Operations

Describe the binary operations

When two numbers are combined according to the instructions given and produce one number we say that they are binary operations. For example, when we add 4 to 6 we get 10, or when we multiply 4 by 3 we get 12. We see that addition of two numbers lead to one number and multiplication of two numbers produce one number. This is binary operation. The instructions can be given either by symbols like X,*,∇and so on or by words.

Performing Binary Operations

Perform binary operations

Example 1

Evaluate:

a. 175×7
b. $(3 \times 205) + (12 \times 75)$
c. $(640 \times 25) - (2025 \div 15\,595)$

solution

a. $175 \times 7 = 1\,225$

b. $(3 \times 205) + (12 \times 75) = 615 + 900$

$= 1\,515$

c. $(640 \times 25) - (2025 \div 5) = 16\,000 - 405$

$= 15\,595$

Example 2

Find

If $a * b = 4a + (2b - 3)$ find $5 * 4$.

solution

We are given that $a * b = 4a + (2b - 3)$

Now, $5 * 4 = 4(5) + (2(4) - 3)$

$= 20 + (8 - 3)$

$= 20 + 5$

$= 25$

Example 3

Solve

If $r \Delta s = rs - \frac{3r}{s}$ find $12 \Delta 4$

Solution

We are given $r \Delta s = rs - \frac{3r}{s}$

$$\text{Now, } 12 \Delta 4 = 12(4) - \frac{3(12)}{4}$$

$$= 48 - 9$$

$$= 39$$

Example 4

evaluate,

If $x * y = \frac{625}{x} - 5y$, find $5 * (25 * 3)$

Solution

We are given $x * y = \frac{625}{x} - 5y$

now, $5 * (25 * 3) = 5 * (\frac{625}{25} - 5(3))$ (do operation inside the brackets first)

$= 5 * 10$

so, $5 * 10 = \frac{625}{5} - 5(10)$

$= 25 - 50$

$= -25$

Example 5

Calculate

Given that $n \Delta m = \frac{3\sqrt{nm}}{\sqrt[3]{m}}$. Find $3 \Delta 27$.

Solution

If $n \Delta m = \frac{3\sqrt{nm}}{\sqrt[3]{m}}$

then, $3 \Delta 27 = \frac{3\sqrt{3 \times 27}}{\sqrt[3]{27}}$

$= \frac{3\sqrt{3 \times 3 \times 3 \times 3}}{\sqrt[3]{3 \times 3 \times 3}}$

$= \frac{3 \times 3 \times 3}{3}$

$= 9$

Brackets in Computation

Brackets are used to group items into brackets and these items inside the brackets are considered as whole. For example,**15 ÷(X + 2) ,**means that x and 2 are added first and their sum should divide 15. If we are given expression with mixed operations, the following order is used to perform the operations: Brackets (B) are opened (O) first followed by Division (D) then Multiplication (M), Addition (A) and lastly Subtraction (S). Shortly is written as BODMAS.

Basic Operations Involving Brackets

Perform basic operations involving brackets

Example 6

Simplify the following expressions:

1. $4 + 2b - (9b \div 3b)$
2. $4z - (2x + z)$

solution

1. $4 + 2b – (9b \div 3b) = 4 + 2b – 3b$ (do division first inside the brackets)
2. $4z – (2x + z) = 4z – 2x – z$ (since the terms inside the brackets are not alike, open the brackets by multiplying each term by the term outside the brackets i.e. - sign)

   $= 4z - z – 2x$ (collect like terms)

   $= 3z – 2x$

Algebraic Expressions Involving the Basic Operations and Brackets

Simplify algebraic expressions involving the basic operations and brackets

Example 7

Evaluate the following expressions:

1. $5x – 2y$ for $x = 4$ and $y = 5$
2. $(3y - 4) + 2x + 1$, for $x = 6$ and $y = 7$
3. $40 - \frac{3}{5}x$, for $x = 25$
4. $(z + 2y) \div x$, for $x = 7, y = 3, z = 15$

solution

1. $5x – 2y = 5(4) – 2(5)$ (we are given $x = 4$ and $y = 5$)

$= 20 – 10$

$= 10$ (subtract)

2. $(3y - 4) + 2x + 1= (3(7) - 4) + 2(6) + 1$(we are given $x =6$, $y= 7$)

$= (21 - 4) + 12 + 1$ (do multiplication first)

$= 17 + 12 + 1$ (do operation inside the bracket first)

$= 30$ (add)

3. $40 - \frac{3}{5}x = 40 - \frac{3}{5}(25)$ (we are given $x = 25$)

$= 40 – 3(5)$ (divide 25 by 5 to remove the fraction)

$= 40 – 15$ (multiply first)

$= 25$ (subtract)

d. $(z + 2y) \div x = (15 + 2(3)) \div 7$ (we are given $x = 7$, $y = 3$, $z = 15$)

$= (15 + 6) \div 7$ (first, multiply)

$= 21 \div 7$ (do operation inside the brackets first)

$= 3$ (divide)

Identities

For example, $3(2y + 3) = 6y + 9$, when $y = 1$, the right hand side (RHS) and the left hand side (LHS) are both equals to 15. If we substitute any other, we obtain the same value on both sides. Therefore the equations which are true for all values of the variables on both sides are called Identities. We can determine whether an equation is an identity or not by showing that an expression on one side is identical to the other expression on the other side.

Example 8

Determine whether or not the following expressions are identities:

1. 4 (a - 2) + 5 = 4a – 3
2. 3(x + 4) = 3x + 12
3. 4(x + 2y - z) = 4x + 8y – 4z
4. 2x – 10 = 0

solution

1. We are given: 4 (a - 2) + 5 = 4a – 3
   consider the LHS
   4 (a - 2) + 5 = 4a – 8 + 5
   = 4a – 3
   therefore, since the LHS = RHS the expression is identity.
2. We are given: 3(x + 4) = 3x + 12
   consider the LHS
   3(x + 4) = 3x + 12 (multiply the terms inside the brackets each by 3)
   since LHS = RHS, then the expression is identity.
3. We are given: 4(x + 2y - z) = 4x + 8y – 4z
   consider the RHS
   4x + 8y – 4z = 4(x + 2y - z)(since 4 is a common factor we fact it out)
   RHS = LHS, therefore the expression is identity.

4. We are given: $2x - 10 = 0$

the value of $x = 5$. We cannot substitute $x = 5$ to determine whether our expression is identity or no
$x = 5$ is the solution to the expression. A good way to determine whether the expression is identity
is to substitute another value of $x \neq 5$. If we substitute on $x = 2$ on LHS we obtain -6. But the value
RHS is 0. So, $-6 \neq 0$. Therefore the expression is not identity

Note the following common identities

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$a^2 - b^2 = (a + b)(a - b)$ (this expression is called difference of two squares.

a and b are real numbers.

Quadratic Expressions

A Quadratic Expression from Two Linear Factors

Form a quadratic expression from two linear factors

A quadratic expression is an expression where the highest exponent of the variable (usually x) is a square (x^2). It is usually written as ax^2+bx+c.

Activity 1

Form a quadratic expression from two linear factors

The General Form of Quadratic Expression

Write the general form of quadratic expression

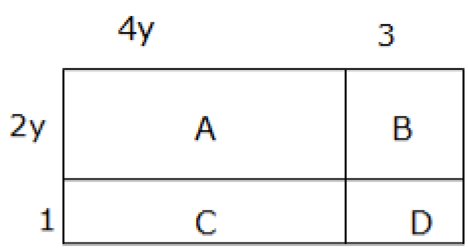
Quadratic expression has the general form of $ax^2 + bx + c$ where $a \neq 0$ and a is a coefficient of x^2 , b is a coefficient of x and c is a constant. its highest power of variable is 2. Examples of quadratic expressions are $2x^2 + x + 1$, $4y^2 + 3$, $3z^2 – 4z + 1$ and so on. In a quadratic expression $3z^2 - 4z + 1$, a = 3, b = -4 and c = 1. Also in quadratic expression $4y^2 + 3$, a = 4, b = 0 and c = 3

Example 9

If you are told to find the area of a rectangle with a length of $4y + 3$ and a width of $2y + 1$.

Solution

Take an example that the rectangle is like the one below:



The area of A = $4y \times 2y = 8y^2$

The area of B = $2y \times 3 = 6y$

The area of C = $4y \times 1 = 4y$

The area of D = $3 \times 1 = 3$

The total area = area of A + area of B + area of C + area of D

$= 8y^2 + 6y + 4y + 1$

$= 8y^2 +10y + 1$

Also we can find the area of a rectangle as follows:

Area = $(4y + 3)\ (2y + 1) = 4y(2y + 1) + 3(2y + 1)$ (each term in the first pair of brackets is multiplied by each term in the second pair of brackets)

$= 8y^2 + 4y + 6y + 3$

$= 8y^2 + 10y +3$

Alternatively

$$\begin{array}{r} 4y + 3 \\ \times\ 2y + 1 \\ \hline 8y^2 + 6y \\ 4y + 3 \\ \hline 8y^2 + 10y + 3 \\ \hline \end{array}$$

The expression $8y^2 + 10y + 3$ is called expanded form of $(4y + 3)(2y + 1)$. $(4y + 3)$ and $(2y + 1)$ are called factors of $8y^2 + 10y + 3$.

Example 10

3x items were bought and each item costs (4x – 3) shillings. Find total amount of money used.

Solution

We have 3x items and each item costs (4x - 3) shillings.

Total cost = $3x(4x - 3)$

$= 12x^2 - 9x$

Therefore the total cost used is $12x^2 - 9x$. This is the expanded form of $3x(4x - 3)$

Factorization

Linear Expressions

Factorize linear expressions

The operation of resolving a quantity into factors, when we expand expressions, is done by removing the brackets. The reverse operation is Factorizing and it is done by adding brackets.

Example 11

Factorize the expression 5a+5b.

Solution

In factorization of 5a+5b, we have to find out a common thing in both terms. We can see that the expression 5a+5b, have got common coefficient in both terms, that is 5. So factoring it out we get 5(a+b).

Example 12

Factorize 18xyz-24xwz

Solution

Factorizing 18xyz-24xwz, we have to find out highest common factor of both terms. Then factor it out, the answer will be 9xz(2y-3w).

Quadratic Expressions

Factorize quadratic expressions

When we write the quadratic expression as a product of two factors we say that we have factorized the expression. We are going to learn two methods used to factorize quadratic expressions. These methods are factorization by Splitting the middle term and factorization by Inspection.

Factorization by splitting the middle term

for example: $6y^2 + 11y + 4$, the term $11y$ can be written as $10y + y$ or $7y + 4y$ or $8y + 3y$ or $6y + 5y$. In order to choose correct terms, consider the coefficients and a constant term. Choose the terms that their product *ac* have the factors whose sum is *b*.

From our example, the product of *ac* is 24. We have to find the terms whose product is 24 and the sum of the factors is 11. These terms are 8 and 3. The product of 8 and 3 is 24 and the sum of 8 and 3 is 11.

Now, we can write $6y^2 + 11y + 4$ as $6y^2 + 3y + 8y + 4$

Which can be written as $3y(2y + 1) + 4(2y + 1)$

Take out common factor which is $2y + 1$ we get, $(2y + 1)(3y + 4)$.

Therefore the expression $6y^2 + 11y + 4$ has the factors which are $(2y + 1)(3y + 4)$.

Example 13

factorize $3x^2 - 2x – 8$ by splitting the middle term.

Solution

Our middle term is $-2x$ and the product of ac is -24. The terms whose product is -24 and the sum of the factors is -2 are 4 and -6.

$3x^2 - 2x - 8 = 3x^2 - 6x + 4x - 8$

$= 3x(x - 2) + 4(x - 2)$

$= (x - 2)(3x + 4)$ (take out common factors)

Example 14

factorize $x^2 + 10x + 25$ by splitting the middle term.

Solution

The middle term is $10x$ and the product of ac is 25. The terms whose product is 25 and the sum of the factors is 10 is 5 and 5.

$x^2 + 10x + 25 = x^2 + 5x + 5x + 25$

$= x(x + 5) + 5(x + 5)$

$= (x + 5)(x + 5)$ (take out common factors)

= (the factors are identical)

If the quadratic expression has two identical factors is called perfect square. The general form of a perfect square is the identity $(a + b)^2 = a^2 + 2ab + b^2$.

Factorization by Inspection

Example 15

factorize $x^2 + 3x + 2$ by inspection.

Solution

Factorization by inspection involves filling in the brackets as follows:

$x^2 + 3x + 2 = (\ \ )(\ \ )$ the first term which is is the product of x and x. And we can put one x at the first bracket and the other x in the second bracket. This will be:

$x^2 + 3x + 2 = (x\ \ )(x\ \ )$ then we have to consider the last term which is 2. 2 is a product of last two constant numbers. These numbers can be 1 and 2, or-1 and -2.

Therefore the possibilities to consider are:

$(x + 1)(x + 2) = + 3x + 2$ (this is the given expression)

$(x - 1)(x - 2) = - 3x + 2$

Note that, the coefficient of x on the RHS is the sum of the last numbers in each pair of brackets. That is, $3 = 1 + 2$; $-3 = -1 + (-2)$.

Therefore the required factors are $(x + 1)$ and $(x + 2)$ i. e. $(x + 1)(x + 2)$.

Example 16

factorize $4x^2 + 5x – 6$ by inspection.

Solution

Since the coefficient of is not 1 then let us find its factors. The first term is $4x^2$ which is the product o and 2x or 4x and x. The constant term is -6, its factors are 1 and -6; -1 and 6; 2 and -3; -2 and 3. We need to use the factors of the first term and the factors of the last term to find the required expression follows:

$(2x + 1)(2x - 6)= 4x^2 - 10x - 6$

$(2x - 1)(2x + 6) = 4x^2 + 10x - 6$

$(2x + 2)(2x - 3) = 4x^2 - 2x - 6$

$(2x - 2)(2x + 3) = 4x^2 + 2x - 6$

$(4x + 1)(x - 6) = 4x^2 - 23x – 6$

$(4x - 6)(x + 1) = 4x^2 - 2x – 6$

$(4x - 1)(x + 6) = 4x^2 + 23x - 6$

$(4x + 6)(x - 1) = 4x^2 + 2x - 6$

$(4x - 2)(x + 3) = 4x^2 + 10x - 6$

$(4x + 3)(x - 2) = 4x^2 - 5x - 6$

$(4x + 2)(x - 3) = 4x^2 - 10x - 6$

$(4x - 3)(x + 2) = 4x^2 + 5x – 6$ (required expression)

Therefore $4x^2 + 5x – 6 = (4x - 3)(x + 2)$.

Exercise 1

Factorization Exercise;

1. Simplify (4a - 6)(2a + 5) – (2a + 5)(4a - 3)

2. Show that $a^2 - b^2 \neq (a-b)^2$.

3. Find the factors of the following quadratic expression by (i) Inspection

(ii) splitting the middle term

a. $3x^2 - x - 10$

b. $x^2 + \frac{4}{3}x + \frac{4}{9}$

c. $x^2 + 3x + 3$

4. Determine whether

(i) $(a+b)^2 = a^2 + b^2$

(ii) $2x + 3 - 4y = 2(x - y) + 3$

(ii) $(3x-1)^2 = 9x^2 + 6x + 1$

Are identities or not.

5. Which of the following expressions are perfect squares?

1. $4x^2 + 12x + 9$
2. $2x^2 + 9x + 6$
3. $16x^2 - 40x + 25$